

On Modules over a G -set

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Abstract

Let R be a commutative ring with unity, M a module over R and let S be a G -set for a finite group G . We define a set MS to be the set of elements expressed as the formal finite sum of the form $\sum_{s \in S} m_s s$ where $m_s \in M$. The set MS is a module over the group ring RG under the addition and the scalar multiplication similar to the RG -module MG defined by Kosan, Lee and Zhou in [9]. With this notion, we not only generalize but also unify the theories of both of the group algebra and the group module, and we also establish some significant properties of $(MS)_{RG}$. In particular, we describe a method for decomposing a given RG -module MS as a direct sum of RG -submodules. Furthermore, we prove the semisimplicity problem of $(MS)_{RG}$ with regard to the properties of M_R , S and G .

1 Introduction

Throughout this paper, G is a finite group with identity element e , R is a commutative ring with unity 1, M is an R -module, RG is the group ring, $H \leq G$ denotes that H is a subgroup of G and S is a G -set with a group action of G on S . If N is an R -submodule of M , it is denoted by $N_R \leq M_R$.

MS denote the set of all formal expression of the form $\sum_{s \in S} m_s s$ where $m_s \in M$ and $m_s = 0$ for almost every s . For elements $\mu = \sum_{s \in S} m_s s$, $\eta = \sum_{s \in S} n_s s \in MS$, by writing $\mu = \eta$ we mean $m_s = n_s$ for all $s \in S$.

We define the sum in MS componentwise

$$\mu + \eta = \sum_{s \in S} (m_s + n_s) s$$

It is clear that MS is an R -module with the sum defined above and the scalar product of $\sum_{s \in S} m_s s$ by $r \in R$ that is $\sum_{s \in S} (rm_s) s$.

For $\rho = \sum_{g \in G} r_g g \in RG$, the scalar product of $\sum_{s \in S} m_s s$ by ρ is

$$\begin{aligned} \rho\mu &= \sum_{s \in S} r_g m_s (sg), \quad sg = s' \in S, \\ &= \sum_{s' \in S} m_{s'} s' \in MS \end{aligned}$$

It is easy to check that MS is a left module over RG , and also as an R -module, it is denoted by $(MS)_{RG}$ and $(MS)_R$, respectively. The RG -module MS is called G -set module of S by

M over RG . It is clear that MS is also a G -set. If S is a G -set and H is a subgroup of G , then S is also an H -set and MS is an RH -module. In addition, if S is a G -set and a group, and $M = R$, then it is easy to verify that RS is a group algebra. On the other hand, if a group acts on itself by multiplication then naturally we have $(MS)_{RG} = (MG)_{RG}$. Since there is a bijective correspondence between the set of actions of G on a set S and the set of homomorphisms from G to Σ_S (Σ_S is the group of permutations on S), the G -set modules is a large class of RG -modules and we would say that $(MG)_{RG}$ introduced in [9] considering the group acting itself by multiplication is a first example of the G -set modules. That is why the notion of the RG -module MS presents a generalization of the structure and discussions of RG -module MG and some principal module-theoretic questions arise out of the structure of $(MS)_{RG}$. Therefore, this new concept generalizes not only the group algebra but also the group module, and also unifies the theory of these two concepts.

The purpose of this paper is to introduce the concept of the RG -module MS , and show the close connection between the properties of $(MS)_{RG}$, M_R , S and G . The semisimplicity of $(MS)_{RG}$ with regard to the properties of M_R , S and G and the decomposition of $(MS)_{RG}$ into RG -submodules will occupy a significant portion of this paper. In Section 1, we present some examples and some properties of $(MS)_{RG}$ to show that an R -module can be extended to RG -modules in various ways via the change of the G -set and the group ring. In Section 2, we give our first major result about the decomposition of a given RG -module MS as a direct sum of RG -submodules. In Section 3, in order to go further into the structure of $(MS)_{RG}$, we first require ε_{MS} that is an extension of the usual augmentation map ε_R and the kernel of ε_{MS} denoted by $\Delta_G(MS)$. Then we give the condition for when $\Delta_G(MS)$ is an RG -submodule of $(MS)_{RG}$. Finally, we are interested in the semisimplicity of $(MS)_{RG}$ according to the properties of M_R , S and G .

We start to set out the idea of G -set modules in more detail by considering some examples of G -set modules and establishing some properties of $(MS)_{RG}$. The following examples for $(MS)_{RG}$ show how useful the notion of G -set module for extension of an R -module M to an RG -module. They also point the relations among G -set S , RG -module MS , G and H where $H \leq G$. Example 1.1 shows that for different group actions on different G -sets of the same finite group we get different extensions of an R -module M to an RG -module. Moreover, we see that these are also RH -modules unsurprisingly in Example 1.2.

Example 1.1 Let M be an R -module, $G = D_6 = \langle a, b : a^3 = b^2 = e, b^{-1}ab = a^{-1} \rangle$ and $r = \sum_{g \in D_6} r_g g = r_1 e + r_2 a + r_3 a^2 + r_4 b + r_5 ba + r_6 ba^2 \in RD_6$.

1. Let $S = G$ and let the group act itself by multiplication. Then $MS = MG$ is an RG -module.
2. Let $S = \{D_6, C_3, C_2, Id\}$ and let G act on its set of subgroups $C_3 = \langle a : a^3 = e \rangle \leq D_6$, $C_2 = \langle b : b^2 = e \rangle \leq D_6$, $Id = \{e\} \leq D_6$ by $g * H = gHg^{-1}$ for $H \leq G$, $g \in G$. Then

$MS = \left\{ \sum_{s \in S} m_s s = m_{Id} Id + m_{C_2} C_2 + m_{C_3} C_3 + m_{D_6} D_6 \mid m_s \in M \right\}$ and we get

$$\begin{aligned} r\mu &= (r_1 m_1 + r_2 m_1 + r_3 m_1 + r_4 m_1 + r_5 m_1 + r_6 m_1) Id \\ &\quad + (r_1 m_{C_2} + r_2 m_{C_2} + r_3 m_{C_2} + r_4 m_{C_2} + r_5 m_{C_2} + r_6 m_{C_2}) C_2 \\ &\quad + (r_1 m_{C_3} + r_2 m_{C_3} + r_3 m_{C_3} + r_4 m_{C_3} + r_5 m_{C_3} + r_6 m_{C_3}) C_3 \\ &\quad + (r_1 m_{D_6} + r_2 m_{D_6} + r_3 m_{D_6} + r_4 m_{D_6} + r_5 m_{D_6} + r_6 m_{D_6}) D_6. \end{aligned}$$

3. Let $S = \{K_1 = \{e, b\}, K_2 = \{a, ba\}, K_3 = \{a^2, ba^2\}\}$ that is the set of right cosets of a fixed subgroup $H = C_2 = \langle b : b^2 = e \rangle \leq D_6$ and let G act on S by $g * (Hx) = H(gx)$ for $x, g \in G$. Then $MS = \left\{ \sum_{s \in S} m_s s = m_{K_1} K_1 + m_{K_2} K_2 + m_{K_3} K_3 \mid m_s \in M \right\}$ and we have the following relations such that

$$\begin{array}{lll} K_1 1 = K_1 & K_2 1 = K_2 & K_3 1 = K_3 \\ K_1 a = K_2 & K_2 a = K_1 & K_3 a = K_1 \\ K_1 a^2 = K_3 & K_2 a^2 = K_3 & K_3 a^2 = K_2 \\ K_1 b = K_1 & K_2 b = K_3 & K_3 b = K_2 \\ K_1 ba = K_2 & K_2 ba = K_1 & K_3 ba = K_3 \\ K_1 ba^2 = K_3 & K_2 ba^2 = K_2 & K_3 ba^2 = K_1. \end{array}$$

So, we get

$$\begin{aligned} r\mu &= (r_1 m_{K_1} + r_4 m_{K_1} + r_3 m_{K_2} + r_5 m_{K_2} + r_2 m_{K_3} + r_6 m_{K_3}) K_1 \\ &\quad + (r_2 m_{K_1} + r_5 m_{K_1} + r_1 m_{K_2} + r_6 m_{K_2} + r_3 m_{K_3} + r_4 m_{K_3}) K_2 \\ &\quad + (r_3 m_{K_1} + r_6 m_{K_1} + r_2 m_{K_2} + r_4 m_{K_2} + r_1 m_{K_3} + r_5 m_{K_3}) K_3. \end{aligned}$$

Example 1.2 Let M be an R -module, $G = D_6 = \langle a, b : a^3 = b^2 = e, b^{-1}ab = a^{-1} \rangle$, $H = C_3 = \langle a : a^3 = e \rangle \leq D_6$ and $k = \sum_{g \in D_6} k_g g = k_1 e + k_2 a + k_3 a^2 \in RC_3$.

1. Let $S = G$ and let the group act itself by multiplication. Then $MS = MG$ is an RH -module.

2. Let $S = \{D_6, C_3, C_2, Id\}$ with the group action defined in Example 1.1 (2). For $\mu = \sum_{s \in S} m_s s = m_{Id} Id + m_{C_2} C_2 + m_{C_3} C_3 + m_{D_6} D_6 \in MS$, we get

$$\begin{aligned} k\mu &= (k_1 m_1 + k_2 m_1 + k_3 m_1) Id + (k_1 m_{C_2} + k_2 m_{C_2} + k_3 m_{C_2}) C_2 \\ &\quad + (k_1 m_{C_3} + k_2 m_{C_3} + k_3 m_{C_3}) C_3 + (k_1 m_{D_6} + k_2 m_{D_6} + k_3 m_{D_6}) D_6. \end{aligned}$$

3. Let $S = \{K_1 = \{e, b\}, K_2 = \{a, ba\}, K_3 = \{a^2, ba^2\}\}$ with the group action defined in Example 1.1 (3). For $\mu = \sum_{s \in S} m_s s = m_{K_1} K_1 + m_{K_2} K_2 + m_{K_3} K_3 \in MS$, we get

$$\begin{aligned} k\mu &= (k_1 m_{K_1} + k_3 m_{K_2} + k_2 m_{K_3}) K_1 + (k_2 m_{K_1} + k_1 m_{K_2} + k_3 m_{K_3}) K_2 \\ &\quad + (k_3 m_{K_1} + k_2 m_{K_2} + k_1 m_{K_3}) K_3 \end{aligned}$$

Now, we make a point of some relations between the R -submodules of M and the RG -submodules of MS by the following results.

Lemma 1.3 *Let N_1, N_2 be R -submodules of M . Then $N_1S + N_2S = MS$ if and only if $N_1 + N_2 = M$.*

Proof Let $N_1S + N_2S = MS$. Take $m \in M$ and so $ms \in MS$ for any $s \in S$. We write $ms = \sum_{s_i \in S} n_{s_i} s_i + \sum_{s_j \in S} n_{s_j} s_j$ for $\sum_{s_i \in S} n_{s_i} s_i \in N_1S$ and $\sum_{s_j \in S} n_{s_j} s_j \in N_2S$ where $n_{s_i} \in N_1, n_{s_j} \in N_2S$. So, there exists i, j such that $m = m_{s_i} + m_{s_j}$.

Let $N_1 + N_2 = M$ and $\mu = \sum_{s \in S} m_s s \in MS$. For all $s \in S$, we can write $m_s = n_s + n'_s$ where $n_s \in N_1, n'_s \in N_2$. Hence, $\mu = \sum_{s \in S} n_s s + \sum_{s \in S} n'_s s$, and so $N_1S + N_2S = MS$. \square

Lemma 1.4 *Let N_1, N_2 be R -submodules of M . Then $N_1S \cap N_2S = 0$ if and only if $N_1 \cap N_2 = 0$.*

Proof Let $N_1S + N_2S = 0$. Take $n \in N_1 \cap N_2$, and so $ns \in N_1S \cap N_2S$. So, $n = 0$ since $ns = 0$.

Conversely, let $N_1 \cap N_2 = 0$. Take $\eta = \sum_{s \in S} n_s s \in N_1S \cap N_2S$. So $n_s \in N_1 \cap N_2$ and $n_s = 0$ for all $s \in S$. Hence, $N_1S \cap N_2S = 0$. \square

From [2] we recall that if G is a finite group, S and T are G -sets, then $\varphi : S \rightarrow T$ is said to be a G -set homomorphism if $\varphi(gs) = g\varphi(s)$ for any $g \in G, s \in S$. If φ is bijective, then φ is a G -set isomorphism. Then we say that S and T are isomorphic G -sets, and we write $S \simeq T$.

For $s \in S, Gs = \{gs : g \in G\}$ is the orbit of s . It is easy to see that Gs is also a G -set under the action induced from that on S . In addition, a subset S' of S is a G -set under the action induced from S if and only if S' is a union of orbits.

Proposition 1.5 *Let M be an R -module, N an R -submodule of M , G a finite group, S a G -set. Then $\frac{MS}{NS} \simeq (\frac{M}{N})S$.*

Proof We know that NS is an RG -submodule of MS . Define a map θ such that

$$\theta : MS \rightarrow (\frac{M}{N})S, \quad \mu = \sum_{s \in S} m_s s \mapsto \theta(\mu) = \sum_{s \in S} (m_s + N)s$$

$$\begin{aligned} \theta(g\mu) &= \theta(g \sum_{s \in S} m_s s) \\ &= g\theta(\mu) \end{aligned}$$

So, θ is a G -set homomorphism. It is clear that θ is a G -set epimorphism. Furthermore, θ is an RG -epimorphism and we get $\ker \theta = NS$. \square

Lemma 1.6 *Any proper subset of an orbit Gs of $s \in S$ is not a G -set under the action induced from S .*

Proof Suppose that a proper subset T of an orbit Gs of $s \in S$ is a G -set. Then there exist $sg \in G, sg \notin T$ for some $g \in G$. Take an element sh in $T, h \in G$, and so

$$\begin{aligned} (gh^{-1})(hs) &= g(h^{-1}(hs)) \\ &= gs \notin T. \end{aligned}$$

Hence, we call the orbit Gs of $s \in S$ the minimal G -set. Moreover, $S = \bigcup_{i \in I} Gs_i$ where I denotes the index of disjoint orbits of S . Hence, we have

$$MS = M\left(\bigcup_{i \in I} Gs_i\right).$$

□

Lemma 1.7 *Let N be an R -submodule of an R -module M , S a G -set. Let I denote the index of disjoint orbits of S , J a subset of I and $S' = \bigcup_{j \in J} Gs_j$ and let Gs_i be an orbit Gs of $s_i \in S$ for $i \in I$. Then we have the following results:*

1. NGs_i is an RG -submodule of MS for $s_i \in S$. Moreover, NGs_i is a minimal RG -submodule of MS containing N under the action induced from that on S .
2. $NS' = N\left(\bigcup_{j \in J} Gs_j\right) = \bigcup_{j \in J} (NGs_j)$.
3. NS' is an RG -submodule of MS .

Proof

1. It is clear that $NGs_i \subseteq MS$. Let $\eta = \sum_{g \in G} n_g gs_i \in NGs_i$, $r \in R$, $h \in G$. Then we have $r\eta \in NGs_i$ and $h\eta = h\left(\sum_{g \in G} n_g gs_i\right) = \sum_{g \in G} n_g hgs_i = \sum_{hg=g' \in G} n_g g' s_i \in NGs_i$. Hence, NGs_i is an RG -submodule of MS . Assume that there is an RG -submodule N_1 of MS such that $N_R \leq (N_1)_{RG} \leq (NGs_i)_{RG}$. Take an element $n \in N$, and so $nhs_i \in N_1$ for some $h \in G$ since $(N_1)_{RG} \leq (NGs_i)_{RG}$. Then $h^{-1}(nhs_i) = (nes_i) = ns_i \in N_1$ and $g(ns_i) = ngs_i \in N_1$ for all $g \in G$. This means that $N_1 = NGs_i$.

2,3. Clear by the definition of MS .

□

Lemma 1.8 *Let L be an RG -submodule of MS , a fixed $s \in S$. Then,*

1. $L_s = \{x \in M \mid \text{there is } y \in L \text{ such that } y = xs + k, k \in MS\}$ is an R -submodule of M .
2. $S_L = \{s \in S \mid \text{there is } x \in M, \text{ and also } k \in L \text{ such that } y = xs + k \in L\}$ is a G -set in S under the action induced from that on S .

Proof

1. It is obvious that L_s is in M . Let $x_1, x_2 \in L_s$ and $r \in R$. Then, there is $y_1 = x_1s + k_1$, $y_2 = x_2s + k_2 \in L$ and $y_1 + y_2 = (x_1 + x_2)s + k_1 + k_2 \in L$ where $x_1 + x_2 \in MS$. Furthermore, $ry_1 = rx_1s + rk_1 \in L$, and so $rx_1 \in L_s$.
2. Let $s \in S'$ and $g, h \in G$. Then $\exists x \in M, \exists k \in L$ such that $y = xs + k \in L$ and

$$xs + k = y = ey = e(xs + k) = xes + ek = xes + k$$

So, $s = es$. Since s is also an element of S , we have

$$\begin{aligned}(hg)y &= (hg)(xs + k) \\ &= (hg)xs + (hg)k.\end{aligned}$$

Hence, we get $(hg)s = h(gs)$.

□

Lemma 1.9 *Let M be an R -module and S a G -set. Let I denote the index of disjoint orbits of S such that $S = \bigcup_{i \in I} Gs_i$ and let Gs_i be an orbit of $s_i \in S$ for $i \in I$. If NGs_i is a simple RG -submodule of MS , then N is a simple R -submodule of M and G is a finite group whose order is invertible in $\text{End}_R(M)$ ($|G|^{-1} \in \text{End}_R(M)$).*

Proof Assume that there is an R -submodule L of M such that $L \leq N \leq M$. Then $(LGs_i)_{RG} \leq (NGs_i)_{RG}$, and by Lemma 1.6 this is a contradiction. So, N is a simple R -submodule of M . □

Theorem 1.10 *Let L be a simple RG -submodule of MS . Then there is a unique simple R -submodule N of M and a unique orbit Gs such that $L = NGs$.*

Proof For some $s \in S$, by Lemma 1.8 L_s is a non-zero R -module. And so, $L_sGs \neq 0$ is an RG -submodule of L . Since L is simple RG -submodule, we have $L_sGs = L$. Then, by Lemma 1.9 L_s is a simple R -submodule of M .

Take an element $s' \in S$ such that $L_{s'}$ is non-zero R -submodule of M . Hence, $L_{s'}Gs' = L = L_sGs$. Take an element $x \in L_{s'}Gs'$. And so, we write

$$x = \sum_{i=1}^n l_i g_i s' = \sum_{i=1}^n k_i g_i s$$

where $l_i \in L_{s'}$, $k_i \in L_s$, $g_i \in G$ and $n = |G|$. Then, there exists $g_j \in G$ such that $g_1 s = g_j s'$, and $s = g_1^{-1} g_j s'$. So, we get $Gs = Gs'$. That is why we can write

$$Gs = S_L = \{s \in S \mid \text{there is } x \in M, \text{ and also } k \in L \text{ such that } y = xs + k \in L\}.$$

Moreover, $N = L_s = L_{s'}$ is unique by the definition of MS . □

On the other hand, the following example shows that the converse of the theorem does not hold.

Example 1.11 *Let $R = \mathbb{Z}_3$, $M = \mathbb{Z}_3$, $G = C_2 = \langle a : a^2 = e \rangle$ and $RG = \mathbb{Z}_3 C_2$. If $S = G$ and G acts on itself by group multiplication then $MS = \mathbb{Z}_3 C_2$ where $\mathbb{Z}_3 C_2$ is semisimple RG -module since $|G| \leq \infty$ and characteristic of R does not divide $|G|$ by Maschke's Theorem. Since $\mathbb{Z}_3 C_2$ is semisimple there is a unique decomposition of $\mathbb{Z}_3 C_2$ by Artin-Wedderburn Theorem. Then, $\mathbb{Z}_3 C_2 \simeq \mathbb{Z}_3 \oplus \mathbb{Z}_3$ as R -module since $|C_2| = 2$. Here, \mathbb{Z}_3 is a simple R -submodule of $\mathbb{Z}_3 C_2$. Moreover, by [11] we have $\mathbb{Z}_3 C_2 \simeq \mathbb{Z}_3 C_2(\frac{1+a}{2}) \oplus \mathbb{Z}_3 C_2(\frac{1-a}{2})$ as RG -module where $\mathbb{Z}_3 C_2(\frac{1+a}{2})$ and $\mathbb{Z}_3 C_2(\frac{1-a}{2})$ are simple RG -submodules of $\mathbb{Z}_3 C_2$. Let $N = \mathbb{Z}_3$ that is a simple R -submodule of M . However, $NGs = \mathbb{Z}_3 C_2$ is not simple RG -module.*

Lemma 1.12 *Let $\{M_i : i \in I\}$ be a family of right R -modules, G a finite group and S a G -set. Then*

$$\left(\left(\bigoplus_{i \in I} M_i \right) S \right)_{RG} = \left(\bigoplus_{i \in I} M_i S \right)_{RG}$$

Proof Consider the following map

$$\left(\bigoplus_{i \in I} M_i \right) S \longrightarrow \bigoplus_{i \in I} M_i S, \quad \sum_{s \in S} (\dots, m_s^{(i)}, \dots) s \longmapsto \sum_{s \in S} (\dots, m_s^{(i)} s, \dots)$$

that is an isomorphism. \square

Theorem 1.13 *An R -module M_R is projective if and only if $(MS)_{RG}$ is projective.*

Proof Assume that M_R is projective. Then for an index I , $(R)^{(I)} \simeq M \oplus A$ where A is a right R -module. So, by Lemma 1.12

$$\begin{aligned} ((RS)^{(I)})_{RG} &\simeq ((R)^{(I)} S)_{RG} \\ &\simeq ((M \oplus A) S)_{RG} \\ &\simeq (MS)_{RG} \oplus (AS)_{RG} \end{aligned}$$

So, $(MS)_{RG}$ is projective.

Now, assume that $(MS)_{RG}$ is projective. Then $((RS)^{(I)})_{RG} \simeq (MS)_{RG} \oplus B$ where B is a right RG -module for some set I . All this concerning modules are also R -modules and $((RS)^{(I)})_R \simeq (MS)_R \oplus B_R$. $((RS)^{(I)})_R$ is a free module because $(RS)_R$ is free. Since $(MS)_R$ is direct summand of a free module, it is projective. So, M_R is projective. \square

2 The Decomposition of $(MS)_{RG}$

The theme of this section is the examination of a G -set module $(MS)_{RG}$ through the study of a decomposition of it. The decompositions of RG and $(MG)_{RG}$ obtained from the idempotent defined as $e_H = \frac{\hat{H}}{|H|}$, where $|H|$ is the order of H and $\hat{H} = \sum_{h \in H} h$, explained in [11] and [15], respectively. A similar method give a criterion for the decomposition of a G -set module $(MS)_{RG}$. In addition, $\text{End}_{RG} MS$ denotes all the RG -endomorphisms of MS .

Lemma 2.1 *Let M be an R -module and H a normal subgroup of finite group G . If $|H|$, the order of H , is invertible in R then $\tilde{e}_H = \frac{\hat{H}}{|H|}$ is an idempotent in $\text{End}_{RG}(MS)$. Moreover, \tilde{e}_H is central in $\text{End}_{RG}(MS)$.*

Proof Firstly, we will show that \tilde{e}_H is an RG -homomorphism. We start with proving that $\hat{H}g = g\hat{H}$ for $g \in G$. Since for all $h_i \in H$, there is $h_{ig} \in H$ such that $h_{ig} = gh_i g^{-1}$, we have that $\hat{H}g = \sum_{h_i \in H} h_i g = \sum_{h_{ig} \in H} gh_{ig} = g\hat{H}$. Therefore, $\frac{\hat{H}}{|H|}rg = rg\frac{\hat{H}}{|H|}$ and we have $\tilde{e}_H(rgm) = rg\tilde{e}_H(m)$ for $m \in MS$, $r \in R$ and $g \in G$. It is also clear that $\tilde{e}_H(m+n) = \tilde{e}_H(m) + \tilde{e}_H(n)$ for $m, n \in MS$, $g \in G$.

Secondly, by using the fact that $\hat{H}.\hat{H} = |H|.\hat{H}$, we get

$$\begin{aligned}\tilde{e}_H(\tilde{e}_H(m)) &= \tilde{e}_H\left(\frac{\hat{H}}{|H|}m\right) \\ &= \tilde{e}_H(m)\end{aligned}$$

So, \tilde{e}_H is an idempotent.

Finally, we prove that \tilde{e}_H is a central idempotent in $\text{End}_{RG}(MS)$. We will show that \tilde{e}_H commutes with every element of $\text{End}_{RG}(MS)$. Let f be in $\text{End}_{RG}(MS)$ and so $\hat{H}f(m) = f(\hat{H}m)$ for $m \in MS$. Thus, we have

$$\begin{aligned}\tilde{e}_H f(m) &= \frac{\hat{H}}{|H|}f(m) \\ &= f\left(\frac{\hat{H}}{|H|}m\right) = f\tilde{e}_H(m).\end{aligned}$$

□

For $\mu = \sum_{g \in G} m_g g \in MG$ and $s_i \in S$, we write

$$\begin{aligned}\mu s_i &= \sum_{g \in G} m_g (gs_i) \\ &= \sum_{gs_i \in S} m_{gs_i} (gs_i) \in MS\end{aligned}$$

Then for $i \in I$ and $\alpha \in M(Gs_i)$, we write $\alpha = \sum_{gs_i \in Gs_i} m_{gs_i} gs_i$. Moreover, we write $\beta = \sum_{i \in I} \sum_{gs_i \in Gs_i} m_{gs_i} gs_i$ for $\beta = \sum_{s \in S} m_s s \in MS$ since $MS = M(\bigcup_{i \in I} Gs_i)$.

Let H be a normal subgroup of G . It is well known that on G/H we have the group action $g(tH) = gtH$ for $g, t \in G$. Consider $g(\sum_{s \in S} m_s(sH)) = (\sum_{s \in S} m_s(gsH))$ for $m_s \in M$.

Let $S' \subset S$ be a G/H -set. Then $S' = \bigcup_{j \in J} G/Hs'_j$ where J denotes the index of disjoint orbits of S' and $MS' = M(\bigcup_{j \in J} G/Hs'_j)$. Then for $\eta = \sum_{s' \in S'} m_{s'} s' \in MS$, we can write $\eta = \sum_{j \in J} \sum_{s' \in G/Hs'_j} m_{s'} s'$.

Hence, we have the following result.

Lemma 2.2 *Let M be an R -module, G a finite group, H a normal subgroup of G , S a G -set and $S' \subset S$ a G/H -set. Then MS' is an RG -module with action defined as $g\eta = g(\sum_{j \in J} \sum_{s' \in G/Hs'_j} m_{s'} s') = g(\sum_{j \in J} \sum_{s' \in G/Hs'_j} m_{s'}(tHs'_j)) = \sum_{j \in J} \sum_{s' \in G/Hs'_j} m_{s'}(gtHs'_j)$ where $\eta = \sum_{j \in J} \sum_{s' \in G/Hs'_j} m_{s'} s' \in MS'$ and $s' = tHs'_j$ for $t \in G$.*

Theorem 2.3 *Let H be a normal subgroup of G , $|H|$ invertible in R and \tilde{e}_H , defined above, then we have $MS = \tilde{e}_H.MS \oplus (1 - \tilde{e}_H).MS$ and there exists a G/H -set $S' \subset S$ such that $\tilde{e}_H.MS \simeq MS'$. More precisely,*

$$\tilde{e}_H.MS = \tilde{e}_H \left(M\left(\bigcup_{i \in I} Gs_i\right) \right) \simeq M\left(\bigcup_{i \in I} \tilde{e}_H Gs_i\right)$$

Proof Firstly, we know that $MG = \tilde{e}_H.MG \oplus (1 - \tilde{e}_H).MG$ and $\tilde{e}_H.MG \simeq M(G/H)$ by the theorem in [15]. Since \tilde{e}_H is a central idempotent by Lemma 2.1, we get $MS = \tilde{e}_H.MS \oplus (1 - \tilde{e}_H).MS$. Now, consider $\theta : G \longrightarrow G.\tilde{e}_H$ where $g \mapsto g\tilde{e}_H$. This is a group homomorphism since $\theta(gh) = gh\tilde{e}_H = gh\tilde{e}_H^2 = g\tilde{e}_H h\tilde{e}_H = \theta(g)\theta(h)$. It is clear that θ is a group epimorphism. We have $\ker\theta = \{g \in G \mid g\tilde{e}_H = \tilde{e}_H\} = \{g \in G \mid (g-1)\tilde{e}_H = 0\} = H$ since $(g-1)\frac{\tilde{H}}{|H|} = 0$ and $g\hat{H} = \hat{H}$ for $g \in H$. Moreover, we get $\frac{G}{\text{er}\theta} = \frac{G}{H} \simeq \text{Im}\theta = G\tilde{e}_H$. So,

$$\tilde{e}_H.MS = \tilde{e}_H \left(M\left(\bigcup_{i \in I} Gs_i\right) \right) = M\left(\bigcup_{i \in I} G\tilde{e}_H s_i\right) \simeq M\left(\bigcup_{i \in I} (G/H)s_i\right)$$

Since $gHs_i = gHs_l$ for $s_i, s_l \in S, i, l \in I$, we get a G/H -set $S' \subset S$ where $\bigcup_{j \in J} (G/H)s_j = S' \subseteq S$.

Hence

$$\tilde{e}_H.MS \simeq M\left(\bigcup_{i \in I} (G/H)s_i\right) = M\left(\bigcup_{j \in J} (G/H)s_j\right) = MS'$$

So, $\tilde{e}_H.MS \simeq MS'$. □

Theorem 2.4 Let M be an R -module and G a finite group. For a G -set $S = \bigcup_{i \in I} Gs_i$ (I denotes the index of disjoint orbits of S), $MS \simeq \bigoplus_{i \in I} MG \setminus \ker \theta_i$ where $\theta_i : MG \longrightarrow MGs_i$ are RG -epimorphisms.

Proof Since $MGs_i \cap MGs_j = \emptyset$ for $i \neq j \in I$ where $S = \bigcup_{i \in I} Gs_i$ and I denotes the index of disjoint orbits of S , we have $MS = M\left(\bigcup_{i \in I} Gs_i\right) = \bigoplus_{i \in I} MGs_i$.

Consider

$$\theta_i : MG \longrightarrow MGs_i, \quad \sum_{g \in G} m_g g \longmapsto \sum_{g \in G} m_g g s_i$$

For $\mu = \sum_{g \in G} m_g g \in MG, r \in R, h \in G$, we have

$$\begin{aligned} \theta_i(r\mu) &= \theta_i\left(r \sum_{g \in G} m_g g\right) = \theta_i\left(\sum_{g \in G} r m_g g\right) = \sum_{g \in G} r m_g g s_i \\ &= r \sum_{g \in G} m_g g s_i = r \theta_i\left(\sum_{g \in G} m_g g\right) = r \theta_i(\mu). \end{aligned}$$

$$\begin{aligned} \theta_i(h\mu) &= \theta_i\left(h \sum_{g \in G} m_g g\right) = \theta_i\left(\sum_{g \in G} m_g h g\right) = \sum_{g \in G} m_g h g s_i \\ &= h \left(\sum_{g \in G} m_g g s_i\right) = h \theta_i\left(\sum_{g \in G} m_g g\right) = h \theta_i(\mu). \end{aligned}$$

Hence, θ_i is an RG -homomorphism. It is clear that θ_i is an epimorphism. Moreover, $MG \setminus \ker \theta_i \simeq \text{Im } \theta_i = MGs_i$. Then,

$$MS = M\left(\bigcup_{i \in I} Gs_i\right) = \bigoplus_{i \in I} MGs_i \simeq \bigoplus_{i \in I} MG \setminus \ker \theta_i.$$

□

3 Augmentation Map on MS and Semisimple G -set Modules

In the theory of the group ring, the augmentation ideal denoted by $\Delta(RG)$ is the kernel of the usual augmentation map ε_R such that

$$\varepsilon_R : RG \longrightarrow R, \quad \sum_{g \in G} r_g g \longmapsto \sum_{g \in G} r_g.$$

The augmentation ideal is always the nontrivial two-sided ideal of the group ring and we have $\Delta(RG) = \left\{ \sum_{g \in G} r_g (g - 1) : r_g \in R, g \in G \right\}$. The augmentation ideal $\Delta(RG)$ is of use for studying not only the relationship between the subgroups of G and the ideals of RG but also the decomposition of RG as direct sum of subrings.

In [9], ε_R is extended to the following homomorphism of R -modules

$$\varepsilon_M : MG \longrightarrow M, \quad \sum_{g \in G} m_g g \longmapsto \sum_{g \in G} m_g.$$

The kernel of ε_M is denoted by $\Delta(MG)$ and

$$\Delta(MG) = \left\{ \sum_{g \in G} m_g (g - 1) : m_g \in M, g \in G \right\}.$$

We devote this section to ε_{MS} that is an extension of ε_M , and to the kernel of ε_{MS} denoted by $\Delta_G(MS)$.

Definition 3.1 *The map*

$$\varepsilon_{MS} : MS \longrightarrow M, \quad \sum_{s \in S} m_s s \longmapsto \sum_{s \in S} m_s$$

is called augmentation map on MS .

In addition, $\varepsilon_{MS}(m_s s_1) = \varepsilon_{MS}(m_s s_2) = m_s$ for $m_s s_1, m_s s_2 \in MS$ where $m_s \in M, s_1, s_2 \in S$, however $m_s s_1 \neq m_s s_2$. Hence, ε_{MS} is not one-to-one.

Lemma 3.2 *Let M be an R -module, G a group and S a G -set. Then $\varepsilon_{MS}(r\mu) = \varepsilon(r) \varepsilon_{MS}(\mu)$ for $\mu = \sum_{s \in S} m_s s \in MS, r = \sum_{g \in G} r_g g \in RG$. In particular, ε_{MS} is an R -homomorphism.*

Proof Let $\mu = \sum_{s \in S} m_s s \in MS, r = \sum_{g \in G} r_g g \in RG$, then

$$\begin{aligned} \varepsilon_{MS}(r\mu) &= \varepsilon_{MS} \left(\sum_{gs \in S} (r_g m_s)(gs) \right) \\ &= \varepsilon_{MS} \left(\sum_{s' \in S} m_{s'} s' \right), \quad m_{s'} = r_g m_s, gs = s' \in S, \\ &= \left(\sum_{g \in G} r_g \right) \left(\sum_{s \in S} m_s \right) \\ &= \varepsilon(r) \varepsilon_{MS}(\mu). \end{aligned}$$

In addition, for $\mu = \sum_{s \in S} m_s s$, $\eta = \sum_{s \in S} n_s s \in MS$, $t \in R$,

$$\begin{aligned} \varepsilon_{MS}(\mu + \eta) &= \varepsilon_{MS}\left(\sum_{s \in S} (m_s + n_s) s\right) \\ &= \sum_{s \in S} m_s + \sum_{s \in S} n_s \\ \varepsilon_{MS}(t\mu) &= \varepsilon_{MS}\left(\sum_{s \in S} (tm_s) s\right) \\ &= t \sum_{s \in S} m_s \end{aligned}$$

□

Furhermore,

$$\ker(\varepsilon_{MS}) = \{\mu = \sum_{s \in S} m_s s \in MS \mid \varepsilon_{MS}(\mu) = \varepsilon_{MS}\left(\sum_{s \in S} m_s s\right) = \sum_{s \in S} m_s = 0\}.$$

It is clear that $\ker(\varepsilon_{MS}) \neq 0$ because for $m_s s_1 + (-m_s s_2) \in MS$, where $m \in M$, $s_1 \neq s_2 \in S$, we have

$$\begin{aligned} \varepsilon_{MS}(m_s s_1 + (-m_s s_2)) &= \varepsilon_{MS}(m_s s_1) + \varepsilon_{MS}(-m_s s_2) \\ &= 0 \end{aligned}$$

Thus, $m_s s_1 + (-m_s s_2) \in \ker(\varepsilon_{MS})$. Moreover, we will characterize the elements of the kernel of ε_{MS} in detail. For this purpose, we define $\Delta_{G,H}(MS) = \left\{ \sum_{h \in H} (h-1)\mu_h \mid \mu_h \in MS \right\}$ where H is a subgroup of finite group G .

Theorem 3.3 *Let M be an R -module, H a subgroup of G , $|H|$ invertible in R , S a G -set and \tilde{e}_H , defined in Lemma 2.1. Then, $\Delta_{G,H}(MS)$ is an RG -module and $\Delta_{G,H}(MS) = (1 - \tilde{e}_H).MS$.*

Proof $\Delta_{G,H}(MS)$ is obviously an RG -module. Now, take any element $\alpha \in \Delta_{G,H}(MS)$. Then we get

$$\begin{aligned} \alpha &= \sum_{h \in H} (h-1)\mu_h \\ &= \sum_{h \in H} (h-1)\left(\sum_{s \in S} m_s s\right) \\ &= \sum_{h \in H} \left(\sum_{s \in S} m_s (h-1)s\right) \\ &= \sum_{h \in H} \left(\sum_{s \in S} m_s (hs - s)\right) \\ &= \sum_{h \in H} \left(\sum_{s \in S} m_s (hs - 1) - (s-1)\right). \end{aligned}$$

On the other hand, for any element $\beta \in (1 - \tilde{e}_H).MS$

$$\begin{aligned}
\beta &= (1 - \tilde{e}_H)\eta \\
&= (1 - \tilde{e}_H)\left(\sum_{s \in S} n_s s\right) \\
&= \left(1 - \frac{\hat{H}}{|H|}\right)\left(\sum_{s \in S} n_s s\right) \\
&= -\frac{1}{|H|}\left(\sum_{h \in H} (h - 1)\right)\left(\sum_{s \in S} n_s s\right) \\
&= \left(\sum_{h \in H} (h - 1)\right)\left(\sum_{s \in S} n'_s s\right) \\
&= \sum_{h \in H} (h - 1)\left(\sum_{s \in S} n'_s s\right) \\
&= \sum_{h \in H} \left(\sum_{s \in S} n'_s (hs - 1) - (s - 1)\right)
\end{aligned}$$

where $\eta \in MS$, $n'_s = -\frac{1}{|H|}n_s$. Hence, $\beta \in \Delta_{G,H}(MS)$. Similarly, $\alpha \in MS.(1 - \tilde{e}_H)$. \square

Furthermore, we write $\Delta_{G,G}(MS) = \Delta_G(MS)$. It is clear that $\ker(\varepsilon_{MS}) = \Delta_G(MS)$ and we have $\ker(\varepsilon_{MS}) = \Delta_G(MS) = (1 - \tilde{e}_G).MS$.

Recall that $\Delta_R(G)$ is the augmentation ideal of RG and for a normal subgroup N of G , $\Delta_R(G, N)$ denote the kernel of the natural epimorphism $RG \rightarrow R(G/N)$ induced by $G \rightarrow G/N$. Moreover, $\Delta_R(G, N)$ is a two-sided ideal of RG generated by $\Delta_R(N)$.

Theorem 3.4 *If N is a normal subgroup of G , then $\Delta_{G,N}(MS) = \Delta_R(N).MS$.*

Proof We know that $\Delta_R(N) = \left\{ \sum_{n \in N} r_n(n - 1) \mid r_n \in R \right\}$ and $\Delta_{G,H}(MS) = \left\{ \sum_{h \in H} (h - 1)\mu_h \mid \mu_h \in MS \right\}$. For $\alpha = \sum_{n \in N} r_n(n - 1) \in \Delta_R(N)$, $\mu = \sum_{s \in S} m_s s \in MS$,

$$\begin{aligned}
\alpha\mu &= \left(\sum_{n \in N} r_n(n - 1) \right) \left(\sum_{s \in S} m_s s \right) \\
&= \sum_{n \in N} r_n(n - 1) \left(\sum_{s \in S} m_s s \right) \\
&= \sum_{n \in N} (n - 1) \left(\sum_{s \in S} (r_n m_s) s \right) \\
&= \sum_{n \in N} (n - 1)\mu_n
\end{aligned}$$

where $\mu_n = \sum_{s \in S} (r_n m_s) s \in MS$. \square

In examination of the studies in group rings which make use of the theory of group modules (see [4], [9], [15]), the semisimplicity problem of the G -set module arises. In [4], the generalized Maschke's Theorem states that a group ring RG is a semisimple Artinian ring if and only if R is a semisimple Artinian ring, G is finite and $|G|^{-1} \in R$. A module theoretic version of the Maschke's Theorem is proven in [9]. This version states that for a nonzero R -module M and a group G , MG is a semisimple module over RG if and only if M is a semisimple module and

G is a finite group whose order is invertible in $\text{End}_R(M)$ that is all the R -endomorphisms of M . The purpose of this section is generalizing the Maschke's Theorem to the G -set modules to give the criterion for the semisimplicity of a G -set module.

Lemma 3.5 *Let M be a nonzero R -module, G a group, S a G -set. If $X \cap \Delta_G(MS) = 0$ for some nonzero RG -submodule X of $(MS)_{RG}$, then G is a finite group.*

Proof Firstly, we know that $\Delta_G(MS)$ is an RG -submodule of $(MS)_{RG}$. Assume that G is an infinite group. Then for any $0 \neq x = m_1 s_1 + \dots + m_k s_k \in X$ where $s_1, \dots, s_k \in S$ are distinct and $m_i s_i \neq 0$, there is an element g of G such that $s_i g \neq s_j$ for $1 \leq i \leq k$. Hence, $(1 - g)x = \sum_{s_i \in S} m_i s_i - \sum_{s_i \in S} m_i g s_i \neq 0$, and also $(1 - g)x \in Y$. On the other hand, $0 \neq (1 - g)x = \sum_{s_i \in S} m_i (s_i - 1) - \sum_{s_i \in S} m_i (g s_i - 1) \in \Delta_G(MS)$. Then, $X \cap \Delta_G(MS) \neq 0$ and this is a contradiction. \square

We recall the following lemma in [10], and also in [9].

Lemma 3.6 [10][9] *Let $X \leq Y$ be right RG -modules and G be a finite group whose order is invertible in $\text{End}_R(V)$. If X is a direct summand of Y as R -modules, then X is a direct summand of Y as RG -modules.*

Theorem 3.7 *Let M be a nonzero R -module, G a group, S a G -set. Then, MS is a semisimple module over RG if and only if M is a semisimple R -module, G is a finite group whose order is invertible in $\text{End}_R(M)$ ($|G|^{-1} \in \text{End}_R(M)$).*

Proof Assume that M is a semisimple R -module, G is a finite group whose order is invertible in $\text{End}_R(M)$. Let Y be an RG -submodule of MS . Firstly, $(MS)_R$ is semisimple since M_R is semisimple. Hence, Y_R is a direct summand of $(MS)_R$. Moreover, $|G|^{-1} \in \text{End}_R(MS)$ since G is finite and $|G|^{-1} \in \text{End}_R(M)$. So, Y_{RG} is a direct summand of $(MS)_{RG}$ by Lemma 3.6 that means $(MS)_{RG}$ is semisimple.

Assume that MS is a semisimple module over RG . $\Delta_G(MS)$ is an RG -submodule of MS and we know that $\Delta_G(MS) \neq MS$. So, $\Delta_G(MS)$ is a proper direct summand of $(MS)_{RG}$. Hence, G is a finite group by Lemma 3.5.

Let N be an R -submodule of M . Then, $(NS)_{RG}$ is an RG -submodule of $(MS)_{RG}$. $(NS)_{RG}$ is a direct summand of $(MS)_{RG}$ because $(MS)_{RG}$ is semisimple, so there is $\alpha^2 = \alpha \in \text{End}_{RG}(MS)$ such that $NS = \alpha(MS)$. Let $\alpha|_M$ be the restriction of α . Consider the composition such that $\gamma : M \xrightarrow{\alpha|_M} MS \xrightarrow{\varepsilon_{MS}} M$, and so $\gamma \in \text{End}_R(M)$. It is clear that $\gamma(M) \subseteq N$. For any $z \in N$, write $z = \alpha(y)$ where $y \in MG$. Then $\gamma(z) = \varepsilon_{MS} \alpha(\alpha(y)) = \varepsilon_{MS} \alpha(y) = \varepsilon_{MS}(z) = z$. Hence, $N = \gamma(M)$, $\gamma(\gamma(z)) = \gamma(z) = z$ and $\gamma^2 = \gamma$ which means that N is a direct summand of M . Therefore, M_R is semisimple R -module.

Assume that $|G|^{-1} \notin \text{End}_R(M)$. Then there is a prime divisor p of $|G|$ such that $p^{-1} \notin \text{End}_R(M)$. We prove that $p : M \rightarrow M$ is not one-to-one. Indeed, if $p : M \rightarrow M$ is one-to-one, then $pM \neq M$ because $p^{-1} \notin \text{End}_R(M)$. $M = pM \oplus Z$ for some nonzero R -submodule Z of M because M_R is semisimple. Since $pM \cap Z = 0$, we get $pZ = 0$. Thus, $p : M \rightarrow M$ is not one-to-one. So, there exists a nonzero direct summand N of M_R such that $pN = 0$ because M_R is semisimple.

Now consider $N\hat{G}$ that is an RG -submodule of $(MS)_{RG}$ and $N\hat{G} \subseteq \Delta_G(NS)$ since $|G|N = 0$. We claim that $\Delta_G(NS)$ is an essential RG -submodule of $(NS)_{RG}$. Let $\sum_{s \in S} n_s s \in NS \setminus \Delta_G(NS)$. Then, $0 \neq \sum_{s \in S} n_s \in N$, and thus $(\sum_{s \in S} n_s s)\hat{G} = (\sum_{s \in S} n_s)\hat{G}$ is a nonzero element of $\Delta_G(NS)$. So $\Delta_G(NS)$ is an essential RG -submodule of $(NS)_{RG}$. Since MS is a semisimple module over RG by hypothesis and $(NS)_{RG}$ is submodule of $(MS)_{RG}$, $(NS)_{RG}$ is semisimple RG -module. Hence, $NS = \Delta_G(NS)$, and so $0 = \varepsilon_{MS}(\Delta_G(NS)) = \varepsilon_{MS}(NS) = N$. This is a contradiction. So, $|G|^{-1} \in \text{End}_R(MS)$. \square

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